A NEW PROOF OF A BISMUT-ZHANG FORMULA FOR SOME CLASS OF REPRESENTATIONS

MAXIM BRAVERMAN† AND BORIS VERTMAN

Abstract. Bismut and Zhang computed the ratio of the Ray-Singer and the combinatorial torsions corresponding to non-unitary representations of the fundamental group. In this note we show that for representations which belong to a connected component containing a unitary representation the Bismut-Zhang formula follows rather easily from the Cheeger-Müller theorem, i.e. from the equality of the two torsions on the set of unitary representations. The proof uses the fact that the refined analytic torsion is a holomorphic function on the space of representations.

1. Introduction

Let M be a closed oriented odd-dimensional manifold and let $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ denote the space of representations of the fundamental group $\pi_1(M)$ of M. For each $\alpha\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, let (E_α,∇_α) be a flat vector bundle over M, whose monodromy representation is equal to α. We denote by $H^\bullet(M,E_\alpha)$ the cohomology of M with coefficients in E_α. Let $\mathrm{Det}(H^\bullet(M,E_\alpha))$ denote the determinant line of $H^\bullet(M,E_\alpha)$.

Reidemeister [21] and Franz [10] used a cell decomposition of M to construct a combinatorial invariant of the representation $\alpha\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, called the *Reidemeister torsion*. In modern language it is a metric on the determinant line $\mathrm{Det}(H^\bullet(M,E_\alpha))$, cf. [19, 2]. If α is unitary, then this metric is independent of the cell decomposition and other choices. In general to define the Reidemeister metric one needs to make some choices. One of such choices is a Morse function $F:M\to\mathbb{R}$. Bismut and Zhang [2] call the metric obtain using the Morse function F the *Milnor metric* and denote it by $\|\cdot\|^{\mathrm{M}}_F$.

Ray and Singer [20] used the de Rham complex to give a different construction of a metric on $\mathrm{Det}(H^\bullet(M,E_\alpha))$. This metric is called the *Ray-Singer metric* and is denoted by $\|\cdot\|^{\mathrm{RS}}$. Ray and Singer conjectured that the Ray-Singer and the Milnor metrics coincide for unitary representation of the fundamental group. This conjecture was proven by Cheeger [8] and Müller [16] and extended by Müller [17] to unimodular representations. For non-unitary representations the two metrics are not equal in general. In the seminal paper [2] Bismut and Zhang computed the ratio of the two metrics using very non-trivial analytic arguments.

In this note we show that for a large class of representations the Bismut-Zhang formula follows quite easily from the original Ray-Singer conjecture. More precisely, let $\alpha_0\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ be a unitary representation which is a regular point of the complex

†Supported in part by the NSF grant DMS-1005888.

analytic set $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ and let $\mathcal{C} \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ denote the connected component of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ which contains α_0. We derive the Bismut-Zhang formula for all representations in $\mathcal{C}$ from the Cheeger-Müller theorem. In other words, we show that knowing that the Milnor and the Ray-Singer metrics coincide on unitary representations one can derive the formula for the ratio of those metrics for all representations in the connected component $\mathcal{C}$.

The proof uses the properties of the refined analytic torsion $\rho_{\mathrm{an}}(\alpha)$ introduced in [3, 6, 5] and of the refined combinatorial torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ introduced in [27, 9]. Both refined torsions are non-vanishing elements of the determinant line $\mathrm{Det}(H^\bullet(M, E_\alpha))$ which depend holomorphically on $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. The ratio of these sections is a holomorphic function

$$\alpha \;\mapsto\; \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$$

on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. We first use the Cheeger-Müller theorem to compute this function for unitary α. Let now $\mathcal{C} \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ be a connected component and suppose that a unitary representation α_0 is a regular point of $\mathcal{C}$. The set of unitary representations can be viewed as the real locus of the connected complex analytic set $\mathcal{C}$. As we know $\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$ for all points of the real locus, we can compute it for all $\alpha \in \mathcal{C}$ by analytic continuation. Since the Ray-Singer norm of $\rho_{\varepsilon,\mathfrak{o}}$ and the Milnor norm of ρ_{an} are easy to compute, we obtain the Bismut-Zhang formula for all $\alpha \in \mathcal{C}$.

The paper is organized as follows. In Section 2, we briefly outline the main steps of the proof. In Subsection 3.5 and Section 3 we recall the construction and some properties of the Milnor metric and of the Farber-Turaev torsion. In Section 4 we recall some properties of the refined analytic torsion. In Section 5 we recall the construction of the holomorphic structure on the determinant line bundle and show that the ratio of the refined analytic and the Farber-Turaev torsions is a holomorphic function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. Finally, in Section 6 we present our new proof of the Bismut-Zhang theorem for representations in the connected component $\mathcal{C}$.

2. The idea of the proof

Our proof of the Bismut-Zhang theorem for representations in the connected component $\mathcal{C}$ consists of several steps. In this section we briefly outline these steps.

Step 1. In [25, 26], Turaev constructed a refined version of the combinatorial torsion associated to an acyclic representation α. Turaev's construction depends on additional combinatorial data, denoted by ε and called the *Euler structure*, as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M, \mathbb{R})$ of M. In [9], Farber and Turaev extended the definition of the Turaev torsion to non-acyclic representations. The Farber-Turaev torsion associated to a representation α, an Euler structure ε, and a cohomological orientation $\mathfrak{o}$ is a non-zero element $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ of the determinant line $\mathrm{Det}\big(H^\bullet(M, E_\alpha)\big)$.

Let us fix a Hermitian metric h^{E_α} on E_α. This scalar product induces a norm $\|\cdot\|^{\mathrm{RS}}$ on $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$, called the *Ray-Singer metric*. In Subsection 3.5 we use the Cheeger-Müller theorem to show that for unitary α

$$\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{RS}} \;=\; 1. \tag{2.1}$$

Remark 2.1. Theorem 10.2 of [9] computes the Ray-Singer norm of $\rho_{\varepsilon,\mathfrak{o}}$ for arbitrary representation $\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, however the proof uses the result of Bismut and Zhang, which we want to prove here for $\alpha \in \mathcal{C}$.

Theorem 1.9 of [5] computes the Ray-Singer metric of $\rho_{\mathrm{an}}(\alpha)$. Combining this result with (2.1) we conclude, cf. Subsection 5.7, that if α is a unitary representation, then

$$\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right| \;=\; \frac{\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}}}{\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{RS}}} \;=\; 1. \tag{2.2}$$

Step 2. The Farber-Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ is a holomorphic section of the determinant line bundle

$$\mathcal{D}et \;:=\; \bigsqcup_{\alpha\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)} \mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$$

over $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. We denote by $\rho_{\mathrm{an}}(\alpha)/\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ the unique complex number such that

$$\rho_{\mathrm{an}}(\alpha) \;=\; \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\cdot\rho_{\varepsilon,\mathfrak{o}}(\alpha) \;\in\; \mathrm{Det}\big(H^\bullet(M,E_\alpha)\big).$$

Since both $\rho_{\varepsilon,\mathfrak{o}}$ and ρ_{an} are holomorphic sections of $\mathcal{D}et$,

$$\alpha \;\mapsto\; \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$$

is a holomorphic function on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$.

Step 3. Let α' denote the representation dual to α with respect to a Hermitian scalar product on $\mathbb{C}^n$. Then the Poincaré duality induces, cf. [9, §2.5] and [5, §10.1], an anti-linear isomorphism[1]

$$D:\;\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big) \;\longrightarrow\; \mathrm{Det}\big(H^\bullet(M,E_{\alpha'})\big).$$

In particular, when α is a unitary representation, D is an anti-linear automorphism of $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$. Hence,

$$\frac{D(\rho_{\mathrm{an}}(\alpha))}{D(\rho_{\varepsilon,\mathfrak{o}}(\alpha))} \;=\; \overline{\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}}. \tag{2.3}$$

[1]There is a sign difference in the definition of the duality operator in [9] and [5], which is not essential for the discussion in this paper.

Using Theorem 7.2 and formula (9.4) of [9] we compute the ratio $D(\rho_{\varepsilon,\mathfrak{o}}(\alpha))/\rho_{\varepsilon,\mathfrak{o}}(\alpha)$, cf. (6.6) (here α is a unitary representation). On the analytic side Theorem 10.3 of [5] computes the ratio $D(\rho_{\rm an}(\alpha))/\rho_{\rm an}(\alpha)$. Combining these two results we get

$$\overline{\frac{\rho_{\rm an}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}} \;=\; f_2(\alpha)\cdot\frac{\rho_{\rm an}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}, \tag{2.4}$$

where f_2 is a function on $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$ computed explicitly in (6.7).

From (2.3) and (2.4) we conclude that

$$\left(\frac{\rho_{\rm an}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right)^2 \;=\; f_1(\alpha)^2\cdot f_2(\alpha) \tag{2.5}$$

for any unitary representation α, cf. (6.9), where $f_1(\alpha)=\rho_{\rm an}(\alpha)/\rho_{\varepsilon,\mathfrak{o}}(\alpha)$.

Step 4. The right hand side of (2.5) is an explicit function of a unitary representation α. It turns out that it is a restriction of a holomorphic function $f(\alpha)$ on $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$ to the set of unitary representations. Recall that the connected component $\mathcal{C}$ contains a regular point which is a unitary representation. The set of unitary representations can be viewed as the *real locus* of the complex analytic set $\mathcal{C}$. Hence any two holomorphic functions which coincide on the set of unitary representations, coincide on $\mathcal{C}$. We conclude now from (2.5) that

$$\left(\frac{\rho_{\rm an}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right)^2 \;=\; f(\alpha), \qquad \text{for all}\ \ \alpha\in\mathcal{C}. \tag{2.6}$$

Step 5. Recall that we denote by $\|\cdot\|^M_F$ the Milnor metric associated to the Morse function F. In Section 3 we compute the Milnor metric

$$\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^M_F \;=\; h_1(\alpha), \tag{2.7}$$

where $h(\alpha)$ is a real valued function on $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$ given explicitly by the right hand side of (3.14)

Theorem 1.9 of [5] computes the Ray-Singer norm of the refined analytic torsion:

$$\|\rho_{\rm an}(\alpha)\|^{\rm RS} \;=\; h_2(\alpha), \tag{2.8}$$

where $h_2(\alpha)$ is a real valued function on $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$ given explicitly by the right hand side of (4.5). Combining (2.6) with (2.8), we get

$$\frac{\|\cdot\|^{\rm RS}}{\|\cdot\|^{\rm M}_F} \;=\; \frac{\|\rho_{\rm an}(\alpha)\|^{\rm RS}}{\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\rm M}_F}\cdot\left|\frac{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}{\rho_{\rm an}(\alpha)}\right| \;=\; \frac{h_2(\alpha)}{h_1(\alpha)\cdot|f(\alpha)|}. \tag{2.9}$$

This is exactly the Bizmut-Zhang formula [2, Theorem 0.2].

The rest of the paper is occupied with the details of the proof outlined above.

3. The Milnor metric and the Farber-Turaev torsion

In this section we briefly recall the definitions and the main properties of the Milnor metric and the Farber-Turaev refined combinatorial torsion. We also compute the Milnor norm of the Farber-Turaev torsion.

3.1. **The Thom-Smale complex.** Set

$$C^k(K,E_\alpha) \;=\; \bigoplus_{\substack{x\in Cr(F)\\ \mathrm{ind}_F(x)=k}} E_{\alpha,x}, \qquad k=1,\dots,n,$$

where $E_{\alpha,x}$ denotes the fiber of E_α over x and the direct sum is over the critical points $x\in Cr(F)$ of the Morse function F with Morse-index $\mathrm{ind}_F(x)=k$. If the Morse function is F generic, then using the gradient flow of F one can define the *Thom-Smale complex* $(C^\bullet(K,E_\alpha),\partial)$ whose cohomology is canonically isomorphic to $H^\bullet(M,E_\alpha)$, cf. for example [2, §I c].

3.2. **The Euler structure.** The *Euler structure* ε on M can be described as (an equivalence class of) a pair (F,c) where c is a 1-chain in M such that

$$\partial c \;=\; \sum_{x\in Cr(F)} (-1)^{\mathrm{ind}_F(x)}\cdot x, \tag{3.1}$$

cf. [7, §3.1]. We denote the set of Euler structures on M by $\mathrm{Eul}(M)$.

Remark 3.3. The Euler structure was introduced by Turaev [26]. Turaev presented several equivalent definitions and the equivalence of these definitions is a nontrivial result. Burghelea and Haller [7] found a very nice way to unify these definitions. They suggested a new definition which is obviously equivalent to the two definitions of Turaev. In this paper we use the definition introduced by Burghelea and Haller.

3.4. **The Kamber-Tondeur form.** To define the Milnor and the Ray-Singer metrics on $\mathrm{Det}(H^\bullet(M,E_\alpha))$ we fix a Hermitian metric h^{E_α} on E_α. This metric is not necessary flat and the measure of non-flatness is given by taking the trace of $(h^{E_\alpha})^{-1}\nabla_\alpha h^{E_\alpha}\in\Omega^1(M,\mathrm{End}E_\alpha)$ which defines the *Kamber-Tondeur form*

$$\theta(h^{E_\alpha}) \;:=\; \mathrm{Tr}\,\big[\,(h^{E_\alpha})^{-1}\nabla_\alpha h^{E_\alpha}\,\big] \;\in\; \Omega^1(M), \tag{3.2}$$

cf. [14] (see also [2, Ch. IV]).

Let $\mathrm{Det}(E_\alpha)\to M$ denote the determinant line bundle of E_α, i.e. the line bundle whose fiber over $x\in M$ is equal to the determinant line $\mathrm{Det}(E_{\alpha,x})$ of the fiber $E_{\alpha,x}$ of E_α. The connection ∇_α and the metric h^{E_α} induce a flat connection $\nabla_\alpha^{\mathrm{Det}}$ and a metric $h^{\mathrm{Det}(E_\alpha)}$ on $\mathrm{Det}(E_\alpha)$. Then

$$\theta(h^{\mathrm{Det}(E_\alpha)}) \;=\; \theta(h^{E_\alpha}). \tag{3.3}$$

For a curve $\gamma:[a,b]\to M$ let

$$\alpha(\gamma):\,E_{\alpha,\gamma(a)}\;\to\;E_{\alpha,\gamma(b)},\quad \alpha^{\mathrm{Det}}(\gamma):\,\mathrm{Det}(E_{\alpha,\gamma(a)})\;\to\;\mathrm{Det}(E_{\alpha,\gamma(b)}) \tag{3.4}$$

denote the parallel transports along γ. Then

$$\mathrm{Det}\,\big(\alpha(\gamma)\big) \;=\; \alpha^{\mathrm{Det}}(\gamma). \tag{3.5}$$

Let $\widetilde{\gamma}(t) \in \mathrm{Det}(E_{\alpha,\gamma(t)})$ denote the horizontal lift of the curve γ. By the definition of the Kamber-Tondeur form we have

$$\log \frac{h^{\mathrm{Det}(E_\alpha)}(\widetilde{\gamma}(b),\widetilde{\gamma}(b))}{h^{\mathrm{Det}(E_\alpha)}(\widetilde{\gamma}(a),\widetilde{\gamma}(a))} \;=\; \int_\gamma \theta(h^{\mathrm{Det}(E_\alpha)}) \;=\; \int_\gamma \theta(h^{E_\alpha}), \tag{3.6}$$

where in the last equality we used (3.3).

If γ is a closed curve, $\gamma(a)=\gamma(b)$, we obtain

$$\frac{h^{\mathrm{Det}(E_\alpha)}(\widetilde{\gamma}(b),\widetilde{\gamma}(b))}{h^{\mathrm{Det}(E_\alpha)}(\widetilde{\gamma}(a),\widetilde{\gamma}(a))} \;=\; \big|\alpha^{\mathrm{Det}}(\gamma)\big|^2 \;=\; \big|\,\mathrm{Det}\,\big(\alpha(\gamma)\big)\big|^2.$$

Hence from (3.6) we obtain

$$\big|\,\mathrm{Det}\,\big(\alpha(\gamma)\big)\big| \;=\; e^{\frac{1}{2}\int_\gamma \theta(h^{E_\alpha})}. \tag{3.7}$$

3.5. **The Milnor metric.** The Hermitian metric h^{E_α} on E_α defines a scalar product on the spaces $C^\bullet(K,E_\alpha)$ and, hence, a metric $\|\cdot\|_{\mathrm{Det}(C^\bullet(K,E_\alpha))}$ on the determinant line of $C^\bullet(K,E_\alpha)$. Using the isomorphism

$$\phi:\ \mathrm{Det}\,\big(\,C^\bullet(K,E_\alpha)\,\big) \longrightarrow \mathrm{Det}\,\big(\,H^\bullet(M,E_\alpha)\,\big), \tag{3.8}$$

cf. formula (2.13) of [5], we thus obtain a metric on $\mathrm{Det}\,\big(\,H^\bullet(M,E_\alpha)\,\big)$, called the *Milnor metric* associated with the Morse function F and denoted by $\|\cdot\|^{\mathrm{M}}_F$.

3.6. **The Farber-Turaev torsion.** Turaev [26] showed that if an Euler structure is fixed, then the scalar product on the spaces $C^k(K,E_\alpha)$ allows one to construct not only a metric on the determinant line $\mathrm{Det}\big(C^\bullet(K,E_\alpha)\big)$ but also an element of this line, defined modulo sign.

We recall briefly Turaev's construction. Fix a base point $x_* \in M$. Then every Euler structure ε can be represented by a pair (F,c) such that

$$c \;=\; \sum_{x\in Cr(F)} (-1)^{\mathrm{ind}_F(x)}\gamma_x,$$

with $\gamma_x:[0,1]\to M$ being a smooth curve such that $\gamma_x(0)=x_*$ and $\gamma_x(1)=x$. The chain c is often referred to as a *Turaev spider*.

We need to construct an element of the the determinant line $\mathrm{Det}\big(C^\bullet(K,E_\alpha)\big)$ of the cochain complex $C^\bullet(K,E_\alpha)$. It is easier to start with constructing an element in the determinant line of the *chain* complex. Since the cochain complex is dual to the chain complex of the bundle $E_{\alpha'}$, where α' denote the representation dual to α, we construct an element in the determinant line $\mathrm{Det}\big(C_\bullet(K,E_{\alpha'})\big)$. This is done as follows:

Fix an element $v_* \in \mathrm{Det}(E_{\alpha',x_*})$ whose norm with respect to the Hermitian metric $h^{\mathrm{Det}(E_{\alpha'})}$ is equal to 1 and set

$$v_x \;:=\; {\alpha'}^{\mathrm{Det}}(\gamma_x)(v_*) \;\in\; \mathrm{Det}(E_{\alpha',x}),$$

where ${\alpha'}^{\mathrm{Det}}$ is the monodromy of the induced connection on the determinant line bundle $\mathrm{Det}(E_{\alpha'})$, cf. (3.4). Let

$$|v|^{\mathrm{Det}(E_{\alpha'})} \;:=\; \sqrt{h^{\mathrm{Det}(E_{\alpha'})}(v,v)}$$

denote the norm induced on $\mathrm{Det}(E_{\alpha'})$ by the Hermitian metric $h^{\mathrm{Det}(E_{\alpha'})}$. Then from (3.6) we obtain

$$|v_x|^{\mathrm{Det}(E_{\alpha'})} \;=\; \frac{|v_x|^{\mathrm{Det}(E_{\alpha'})}}{|v_*|^{\mathrm{Det}(E_{\alpha'})}} \;=\; e^{\frac{1}{2}\int_{\gamma_x}\theta(h^{\mathrm{Det}(E_{\alpha'})})} \;=\; e^{-\frac{1}{2}\int_{\gamma_x}\theta(h^{\mathrm{Det}(E_{\alpha})})}. \tag{3.9}$$

Let

$$v \;=\; \prod_{x\in Cr(F)} v_x^{(-1)^{\mathrm{ind}_F(x)}} \in\; \mathrm{Det}\left(C_\bullet(K,E_{\alpha'})\right)/\pm\,.$$

(The sign indeterminacy comes form the choice of the order of the critical points of F.) From (3.9) we conclude that

$$\|v\|_{\mathrm{Det}(C_\bullet(K,E_{\alpha'}))} \;=\; e^{-\frac{1}{2}\int_c\theta(h^{\mathrm{Det}(E_\alpha)})}. \tag{3.10}$$

Let $\langle\cdot,\cdot\rangle$ denote the natural pairing

$$\mathrm{Det}\left(C^\bullet(K,E_\alpha)\right)\times\mathrm{Det}\left(C_\bullet(K,E_{\alpha'})\right) \;\to\; \mathbb{C}$$

and let $\nu\in\mathrm{Det}\left(C^\bullet(K,E_\alpha)\right)/\pm$ be the unique element such that $\langle\nu,v\rangle=1$. From (3.10) we now obtain

$$\|\nu\|_{\mathrm{Det}(C^\bullet(K,E_\alpha))} \;=\; e^{\frac{1}{2}\int_c\theta(h^{\mathrm{Det}(E_\alpha)})}. \tag{3.11}$$

Using the isomorphism (3.8) we obtain an element

$$\phi(\nu) \;\in\; \mathrm{Det}\left(H^\bullet(M,E_\alpha)\right)/\pm\,. \tag{3.12}$$

To fix the sign one can choose a *cohomological orientation* $\mathfrak{o}$, i.e. an orientation of the determinant line $\mathrm{Det}(H^\bullet(M,\mathbb{R})$. Thus, given the Euler structure ε and the cohomological orientation $\mathfrak{o}$ we obtain a sign refined version of $\phi(\nu)$ which we call the *Farber-Turaev torsion* and denote by

$$\rho_{\varepsilon,\mathfrak{o}}(\alpha)\in\;\mathrm{Det}\left(H^\bullet(M,E_\alpha)\right). \tag{3.13}$$

3.7. **The Milnor norm of the Farber-Turaev torsion.** From (3.11) we immediately get

$$\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{M}}_F \;=\; e^{\frac{1}{2}\int_c\theta(h^{E_\alpha})}. \tag{3.14}$$

In particular, if α is a unitary representation, then h^{E_α} is a flat Hermitian metric and $\theta(h^{E_\alpha})=0$. Hence, if α is unitary, then

$$\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{M}}_F \;=\; 1. \tag{3.15}$$

We now use the Cheeger-Müller theorem to conclude that

$$\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{RS}} \;=\; 1, \qquad \text{if } \alpha \text{ is unitary.} \tag{3.16}$$

3.8. **Dependence of the Farber-Turaev torsion on the Euler structure.** For a homology class $h \in H_1(M,\mathbb{Z})$ and an Euler structure $\varepsilon = (F,c) \in \mathrm{Eul}(M)$ we set

$$h\varepsilon \;:=\; (F, c+h) \;\in\; \mathrm{Eul}(M). \tag{3.17}$$

This defines a free and transitive action of $H_1(M,\mathbb{Z})$ on $\mathrm{Eul}(M)$, cf. [9, §5] or [7, §3.1].

One easily checks, cf. [9, page 211], that

$$\rho_{h\varepsilon,\mathfrak{o}}(\alpha) \;=\; \mathrm{Det}\,\big(\alpha(h)\big)\cdot \rho_{\varepsilon,\mathfrak{o}}(\alpha). \tag{3.18}$$

From (3.7) and (3.14) we now obtain

$$\|\rho_{h\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{M}}_{F} \;=\; e^{-\frac12\int_{c+h}\theta(h^{E_\alpha})}. \tag{3.19}$$

4. The Ray-Singer norm of the Refined Analytic Torsion

In [5] Braverman and Kappeler defined an element of $\mathrm{Det}(H^\bullet(M,E_\alpha))$ called the *refined analytic torsion* and denoted by $\rho_{\mathrm{an}}(\alpha)$. They also computed the Ray-Singer norm $\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}}$ of the refined analytic torsion. In this section we recall the result of this computation.

4.1. **The odd signature operator.** Fix a Riemannian metric g^M on M and let $*:\Omega^\bullet(M,E_\alpha)\to\Omega^{m-\bullet}(M,E_\alpha)$ denote the Hodge $*$-operator, where $m=\dim M$. Define the *chirality operator*

$$\Gamma \;=\; \Gamma(g^M):\,\Omega^\bullet(M,E_\alpha)\;\to\;\Omega^\bullet(M,E_\alpha)$$

by the formula

$$\Gamma\,\omega \;:=\; i^r\,(-1)^{\frac{k(k+1)}{2}}\,*\,\omega, \qquad \omega\in\Omega^k(M,E), \tag{4.1}$$

where $r=\frac{m+1}{2}$. The numerical factor in (4.1) has been chosen so that $\Gamma^2=1$, cf. Proposition 3.58 of [1].

The *odd signature operator* is the operator

$$\mathcal{B} \;=\; \mathcal{B}(\nabla_\alpha,g^M) \;:=\; \Gamma\,\nabla_\alpha \;+\; \nabla_\alpha\,\Gamma:\,\Omega^\bullet(M,E_\alpha)\;\longrightarrow\;\Omega^\bullet(M,E_\alpha). \tag{4.2}$$

4.2. **The eta invariant.** We recall from [5, §3] the definition of the sign-refined η-invariant $\eta(\nabla_\alpha,g^M)$ of the (not necessarily unitary) connection ∇_α.

Let $\Pi_>$ (resp. $\Pi_<$) be the projection whose image contains the span of all generalized eigenvectors of $\mathcal{B}$ corresponding to eigenvalues λ with $\mathrm{Re}\,\lambda>0$ (resp. with $\mathrm{Re}\,\lambda<0$) and whose kernel contains the span of all generalized eigenvectors of $\mathcal{B}$ corresponding to eigenvalues λ with $\mathrm{Re}\,\lambda\le 0$ (resp. with $\mathrm{Re}\,\lambda\ge 0$), cf. [18, Appendix B]. We define the η-function of $\mathcal{B}$ by the formula

$$\eta_\theta(s,\mathcal{B}) \;=\; \mathrm{Tr}\,\big[\Pi_>\mathcal{B}^s_\theta\big] \;-\; \mathrm{Tr}\,\big[\Pi_<(-\mathcal{B})^s_\theta\big], \tag{4.3}$$

where θ is an Agmon angle for both operators $\mathcal{B}$ and $-\mathcal{B}$ and $\mathcal{B}^s_\theta$ denotes the complex power of $\mathcal{B}$ defined relative to the spectral cut along the ray $\{re^{i\theta}: r>0\}$, cf. [22, 24]. It was shown by Gilkey, [11], that $\eta_\theta(s,\mathcal{B})$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at $s=0$. Moreover, the number $\eta_\theta(0,\mathcal{B})$ is independent of the Agmon angle θ.

Let $m_+(\mathcal{B})$ (resp., $m_-(\mathcal{B})$) denote the number of eigenvalues of $\mathcal{B}$, counted with their algebraic multiplicities, on the positive (resp., negative) part of the imaginary axis. Let $m_0(\mathcal{B})$ denote algebraic multiplicity of 0 as an eigenvalue of $\mathcal{B}$.

Definition 4.3. *The η-invariant $\eta(\nabla_\alpha, g^M)$ of the pair (∇_α, g^M) is defined by the formula*

$$\eta(\nabla_\alpha, g^M) \;=\; \frac{\eta_\theta(0,\mathcal{B}) + m_+(\mathcal{B}) - m_-(\mathcal{B}) + m_0(\mathcal{B})}{2}. \tag{4.4}$$

If the representation α is unitary, then the operator $\mathcal{B}$ is self-adjoint and $\eta(\nabla_\alpha, g^M)$ is real. If α is not unitary then, in general, $\eta(\nabla_\alpha, g^M)$ is a complex number. Notice, however, that while the real part of $\eta(\nabla_\alpha, g^M)$ is a non-local spectral invariant, the imaginary part $\operatorname{Im}\eta(\nabla_\alpha, g^M)$ of $\eta(\nabla_\alpha, g^M)$ is local and relatively easy to compute, cf. [11, 15].

We also note that the imaginary part of the η-invariant is independent of the Riemannian metric g^M.

4.4. **The Ray-Singer norm of the refine analytic torsion.** Let $\eta(\nabla_\alpha, g^M)$ denote the η-invariant of the odd signature operator corresponding to the connection ∇_α. By Theorem 1.9 of [5]

$$\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}} \;=\; e^{\pi\,\operatorname{Im}(\eta(\nabla_\alpha, g^M))}. \tag{4.5}$$

In particular, when α is a unitary representation, $\eta(\nabla_\alpha, g^M)$ is real and we get

$$\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}} \;=\; 1. \tag{4.6}$$

5. The Determinant Line Bundle over the Space of Representations

The space $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$ of complex n-dimensional representations of $\pi_1(M)$ has a natural structure of a complex analytic space, cf., for example, [6, §13.6]. The disjoint union

$$\mathcal{D}et \;:=\; \bigsqcup_{\alpha\in\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)} \operatorname{Det}\big(H^\bullet(M,E)\big) \tag{5.1}$$

is a line bundle over $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$, called the *determinant line bundle*. In [4, §3], Braverman and Kappeler constructed a natural holomorphic structure on $\mathcal{D}et$, with respect to which both the refined analytic torsion $\rho_{\mathrm{an}}(\alpha)$ and the Farber-Tureav torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ are holomorphic sections. In this section we first recall this construction and then consider the ratio $\rho_{\mathrm{an}}/\rho_{\varepsilon,\mathfrak{o}}$ of these two sections as a holomorphic function on $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$.

5.1. **The flat vector bundle induced by a representation.** Denote by $\pi : \widetilde{M} \to M$ the universal cover of M. For $\alpha \in \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$, we denote by

$$E_\alpha \;:=\; \widetilde{M} \times_\alpha \mathbb{C}^n \;\longrightarrow M \tag{5.2}$$

the flat vector bundle induced by α. Let ∇_α be the flat connection on E_α induced from the trivial connection on $\widetilde{M} \times \mathbb{C}^n$.

For each connected component (in classical topology) $\mathcal{C}$ of $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$, all the bundles E_α, $\alpha \in \mathcal{C}$, are isomorphic, see e.g. [12].

5.2. **The combinatorial cochain complex.** Fix a CW-decomposition $K=\{e_1,\dots,e_N\}$ of M. For each $j=1,\dots,N$, fix a lift $\widetilde{e}_j$, i.e., a cell of the CW-decomposition of $\widetilde{M}$, such that $\pi(\widetilde{e}_j)=e_j$. By (5.2), the pull-back of the bundle E_α to $\widetilde{M}$ is the trivial bundle $\widetilde{M}\times\mathbb{C}^n\to\widetilde{M}$. Hence, the choice of the cells $\widetilde{e}_1,\dots,\widetilde{e}_N$ identifies the cochain complex $C^\bullet(K,\alpha)$ of the CW-complex K with coefficients in E_α with the complex

$$0\ \to\mathbb{C}^{n\cdot k_0}\ \xrightarrow{\partial_0(\alpha)}\ \mathbb{C}^{n\cdot k_1}\ \xrightarrow{\partial_1(\alpha)}\ \cdots\ \xrightarrow{\partial_{m-1}(\alpha)}\ \mathbb{C}^{n\cdot k_m}\ \to\ 0, \tag{5.3}$$

where $k_j\in\mathbb{Z}_{\geq 0}$ $(j=0,\dots,m)$ is equal to the number of j-dimensional cells of K and the differentials $\partial_j(\alpha)$ are $(nk_j\times nk_{j-1})$-matrices depending analytically on $\alpha\in\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$.

The cohomology of the complex (5.3) is canonically isomorphic to $H^\bullet(M,E_\alpha)$. Let

$$\phi_{C^\bullet(K,\alpha)}:\ \operatorname{Det}\big(\,C^\bullet(K,\alpha)\,\big)\longrightarrow\ \operatorname{Det}\big(\,H^\bullet(M,E_\alpha)\,\big) \tag{5.4}$$

denote the natural isomorphism between the determinant line of the complex and the determinant line of its cohomology, cf. [5, §2.4]

5.3. **The holomorphic structure on $\mathcal{D}et$.** The standard bases of $\mathbb{C}^{n\cdot k_j}$ $(j=0,\dots,m)$ define an element $c\in\operatorname{Det}\big(C^\bullet(K,\alpha)\big)$, and, hence, an isomorphism

$$\psi_\alpha:\mathbb{C}\longrightarrow\ \operatorname{Det}\big(C^\bullet(K,\alpha)\big),\qquad z\ \mapsto z\cdot c.$$

Then the map

$$\sigma:\ \alpha\ \mapsto\ \phi_{C^\bullet(K,\alpha)}\big(\,\psi_\alpha(1)\,\big)\ \in\ \operatorname{Det}\big(\,H^\bullet(M,E_\alpha)\,\big), \tag{5.5}$$

where $\alpha\in\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$ is a nowhere vanishing section of the determinant line bundle $\mathcal{D}et$ over $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$.

Definition 5.4. *We say that a section $s(\alpha)$ of $\mathcal{D}et$ is* holomorphic *if there exists a holomorphic function $f(\alpha)$ on $\operatorname{Rep}(\pi_1(M),\mathbb{C}^n)$, such that $s(\alpha)=f(\alpha)\cdot\sigma(\alpha)$.*

This defines a holomorphic structure on $\mathcal{D}et$, which is independent of the choice of the lifts $\widetilde{e}_1,\dots,\widetilde{e}_N$ of $e_1,\dots,e_N$, since for a different choice of lifts the section $\sigma(\alpha)$ will be multiplied by a constant. It is shown in [4, §3.5] that this holomorphic structure is also independent of the CW-decomposition K of M.

Theorem 5.5. *Both the refined analytic torsion $\rho_{\rm an}(\alpha)$ and the Farber-Turaev torsion $\rho_{\varepsilon,\mathfrak{o}}(\alpha)$ are holomorphic sections of $\mathcal{D}et$ with respect to the holomorphic structure described above.*

Proof. The fact that the Farber-Turaev torsion is holomorphic is established in Proposition 3.7 of [4]. The fact that the refined analytic torsion is holomorphic is proven in Theorem 4.1 of [4]. □

5.6. **The ratio of the torsions as a holomorphic function.** Since both $\rho_{\varepsilon,\mathfrak{o}}$ and ρ_{an} are holomorphic nowhere vanishing section of the same line bundle there exists a holomorphic function

$$\kappa : \mathrm{Rep}(\pi_1(M), \mathbb{C}^n) \;\to\; \mathbb{C}\backslash\{0\}$$

such that

$$\rho_{\mathrm{an}}(\alpha) \;=\; \kappa(\alpha)\cdot\rho_{\varepsilon,\mathfrak{o}}(\alpha).$$

We shall denote this function by

$$\kappa(\alpha) = \frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}. \tag{5.6}$$

5.7. **The absolute value of $\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}$ for unitary representations.** Combining (4.6) with (3.16) we obtain

$$\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right| \;=\; \frac{\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}}}{\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{RS}}} \;=\; 1, \qquad \text{if } \alpha \text{ is unitary.} \tag{5.7}$$

6. The Bismut-Zhang theorem for some non-unitary representations

We now present our proof of the Bismut-Zhang theorem [2, Theorem 0.2] for representations in the connected component $\mathcal{C}$.

6.1. **The duality operator.** Let α' denotes the representation dual to α. The Poincaré duality defines a non-degenerate pairing

$$\mathrm{Det}\left(H^k(M,E_\alpha\right)\times\mathrm{Det}\left(H^{m-k}(M,E_{\alpha'})\right) \;\to \mathbb{C}, \qquad k=0,\dots,m,$$

and, hence, an anti-linear map

$$D : \mathrm{Det}\left(H^\bullet(M,E_\alpha)\right) \;\to\; \mathrm{Det}\left(H^\bullet(M,E_{\alpha'})\right) \tag{6.1}$$

see [9, §2.5] and [5, §10.1] for details.

By Theorem 10.3 of [5] we have

$$D\,\rho_{\mathrm{an}}(\alpha) \;=\; \rho_{\mathrm{an}}(\alpha')\cdot e^{2i\pi\left(\eta(\nabla_\alpha,g^M)-(\mathrm{rank}\,E)\,\eta_{\mathrm{trivial}}(g^M)\right)}, \tag{6.2}$$

where $\eta(\nabla_\alpha, g^M)$ is defined in Definition 4.3 and η_{trivial} is the η-invariant corresponding to the standard connection on the trivial line bundle $M\times\mathbb{C}\to M$.

6.2. **The dual of the Farber-Turaev torsion.** By Theorem 7.2 of [9]

$$D\,\rho_{\varepsilon,\mathfrak{o}}(\alpha) \;=\; \pm\,\rho_{\varepsilon^*,\mathfrak{o}}(\alpha'), \tag{6.3}$$

where $\varepsilon^* := (-F,-c)$ is the *dual Euler structure* on M.

We shall use formula (3.18) in the following situation: if $\varepsilon = (F,c)\in\mathrm{Eul}(M)$ then the Euler structure $\varepsilon^* := (-F,-c)$ is called *dual* to ε. Since $H_1(M,\mathbb{Z})$ acts freely and transitively on $\mathrm{Eul}(M)$ there exists $c_\varepsilon\in H_1(M,\mathbb{Z})$ such that

$$\varepsilon \;=\; c_\varepsilon\,\varepsilon^*. \tag{6.4}$$

The homology class c_ε was introduced by Turaev [26] and is called the *characteristic class of the Euler structure*. From (3.18) and (6.3) we now conclude that

$$D\,\rho_{\varepsilon,\mathfrak{o}}(\alpha) \;=\; \pm\rho_{\varepsilon^*,\mathfrak{o}}(\alpha') \;=\; \pm\,\mathrm{Det}\,\big(\alpha'(c_\varepsilon)\big)\cdot\rho_{\varepsilon,\mathfrak{o}}(\alpha'). \tag{6.5}$$

If α is a unitary representation, then $\alpha=\alpha'$. Hence, it follows from (6.5) that

$$\rho_{\varepsilon,\mathfrak{o}}(\alpha') \;=\; \pm\big(\mathrm{Det}\,\big(\alpha(c_\varepsilon)\big)\big)^{-1}\cdot D\,\rho_{\varepsilon,\mathfrak{o}}(\alpha). \tag{6.6}$$

6.3. **The ratio of torsions for unitary representations.** Combining (6.2) and (6.6) we conclude that for unitary α

$$\frac{D\rho_{\mathrm{an}}(\alpha)}{D\rho_{\varepsilon,\mathfrak{o}}(\alpha)} \;=\; \pm\,\mathrm{Det}\,\big(\alpha(c_\varepsilon)\big)\cdot e^{2i\pi\,\left(\eta(\nabla_\alpha,g^M)-(\mathrm{rank}\,E)\,\eta_{\mathrm{trivial}}(g^M)\right)}\cdot\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}. \tag{6.7}$$

Since D is an anti-linear involution we have

$$\frac{D\rho_{\mathrm{an}}(\alpha)}{D\rho_{\varepsilon,\mathfrak{o}}(\alpha)} \;=\; \overline{\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}}.$$

Hence, it follows from (6.7) that

$$\left(\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right)^2 \;=\; \pm\,\mathrm{Det}\,\big(\alpha(c_\varepsilon)\big)\cdot e^{-2i\pi\,\left(\eta(\nabla_\alpha,g^M)-(\mathrm{rank}\,E)\,\eta_{\mathrm{trivial}}(g^M)\right)}\cdot\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right|^2. \tag{6.8}$$

Combining this equality with (5.7) we obtain for unitary α

$$\left(\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}\right)^2 \;=\; \pm\,\mathrm{Det}\,\big(\alpha(c_\varepsilon)\cdot e^{-2i\pi\,\left(\eta(\nabla_\alpha,g^M)-(\mathrm{rank}\,E)\,\eta_{\mathrm{trivial}}(g^M)\right)}. \tag{6.9}$$

6.4. **The ratio of torsions for non-unitary representations.** Suppose now that $\mathcal{C}\subset\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ is a connected component and $\alpha_0\subset\mathcal{C}$ is a unitary representation which is a regular point of the complex analytic set $\mathcal{C}$. The set of unitary representations is the fixed point set of the anti-holomorphic involution

$$\tau:\,\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)\;\to\mathrm{Rep}(\pi_1(M),\mathbb{C}^n),\qquad \tau:\,\alpha\;\mapsto\;\alpha'.$$

Hence it is a totally real submanifold of $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ whose real dimension is equal to $\dim_{\mathbb{C}}\mathcal{C}$, see for example [13, Proposition 3]. In particular there is a holomorphic coordinates system $(z_1,\dots,z_r)$ near α_0 such that the unitary representations form a *real neighborhood* of α_0, i.e. the set $\mathrm{Im}\,z_1=\ldots=\mathrm{Im}\,z_r=0$. Therefore, cf. [23, p. 21], if two holomorphic functions coincide on the set of unitary representations they also coincide on $\mathcal{C}$. We conclude that the equation (6.9) holds for all representations $\alpha\in\mathcal{C}$. Hence, using (4.5) and (3.14) we obtain for every $\alpha\in\mathcal{C}$

$$\frac{\|\cdot\|^{\mathrm{RS}}}{\|\cdot\|^{\mathrm{M}}_F} \;=\; \frac{\|\rho_{\mathrm{an}}(\alpha)\|^{\mathrm{RS}}}{\|\rho_{\varepsilon,\mathfrak{o}}(\alpha)\|^{\mathrm{M}}_F}\cdot\left|\frac{\rho_{\varepsilon,\mathfrak{o}}(\alpha)}{\rho_{\mathrm{an}}(\alpha)}\right| \;=\; \big|\,\mathrm{Det}\,\big(\alpha(c_\varepsilon)\big)\big|^{-1/2}\cdot e^{-\frac{1}{2}\int_c\theta(h^{E_\alpha})}. \tag{6.10}$$

6.5. **The absolute value of the determinant of $\alpha(c_\varepsilon)$.** Let

$$\mathrm{PD}:\ H_1(M,\mathbb{R})\ \to\ H^{n-1}(M,\mathbb{R})$$

denote the Poincaré isomorphism. By Proposition 3.9 of [7] there exists a map

$$P:\mathrm{Eul}(M)\ \to\ \Omega^{n-1}(M,\mathbb{R})$$

such that

$$\begin{aligned} P(h\varepsilon)\ &=\ P(\varepsilon)\ +\ \mathrm{PD}(h),\\ P(\varepsilon^*)\ &=\ -P(\varepsilon), \end{aligned} \tag{6.11}$$

and if $\varepsilon=(X,c)$ then for every $\omega\in\Omega^1(M,\mathbb{R})$

$$\int_c \omega\ =\ \int_M \omega\wedge X^*\Psi(g)\ -\ \int_M \omega\wedge P(\varepsilon). \tag{6.12}$$

Here $\Psi(g)$ is the Mathai-Quillen current on TM, cf. [2, §III c] and $X^*\Psi(g)$ denotes the pull-back of this current by $X:M\to TM$.

Combining (6.4) with (6.11) we obtain

$$P(\varepsilon)\ =\ P(\varepsilon^*)\ +\ \mathrm{PD}(c_\varepsilon)\ =\ -P(\varepsilon)\ +\ \mathrm{PD}(c_\varepsilon).$$

Thus

$$\mathrm{PD}(c_\varepsilon)\ =\ 2\,P(\varepsilon). \tag{6.13}$$

Combining this equality with (6.12) we get

$$\int_c \omega\ =\ \int_M \omega\wedge X^*\Psi(g)\ -\ \frac{1}{2}\int_M \omega\wedge \mathrm{PD}(c_\varepsilon). \tag{6.14}$$

Notice now that

$$\int_M \omega\wedge\mathrm{PD}(c_\varepsilon)\ =\ \int_{c_\varepsilon}\omega.$$

Hence, from (6.14) we obtain

$$\int_{c_\varepsilon}\omega\ =\ -2\int_c\omega\ +\ 2\int_M \omega\wedge X^*\Psi(g). \tag{6.15}$$

In particular, setting $\omega=\theta(h^{E_\alpha})$ and using (3.7) we obtain

$$\big|\,\mathrm{Det}\,\big(\alpha(c_\varepsilon)\big)\big|\ =\ e^{-\int_c\theta(h^{E_\alpha})+\int_M\theta(h^{E_\alpha})\wedge X^*\Psi(g)}. \tag{6.16}$$

Combining this equality with (6.10)

$$\frac{\|\cdot\|^{\mathrm{RS}}}{\|\cdot\|^{\mathrm{M}}_F}\ =\ e^{-\frac{1}{2}\int_M\theta(h^{E_\alpha})\wedge X^*\Psi(g)}, \tag{6.17}$$

which is exactly the Bizmut-Zhang formula [2, Theorem 0.2].

References

[1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, 1992.

[2] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205** (1992).

[3] M. Braverman and T. Kappeler, *A refinement of the Ray-Singer torsion*, C.R. Acad. Sci. Paris **341** (2005), 497–502.

[4] ———, *Ray-Singer type theorem for the refined analytic torsion*, **243** (2007), 232–256.

[5] ———, *Refined Analytic Torsion as an Element of the Determinant Line*, Geometry & Topology **11** (2007), 139–213.

[6] ———, *Refined Analytic Torsion*, J. Differential Geom. **78** (2008), no. 1, 193–267.

[7] D. Burghelea and S. Haller, *Euler Structures, the Variety of Representations and the Milnor-Turaev Torsion*, Geom. Topol. **10** (2006), 1185–1238.

[8] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. **109** (1979), 259–300.

[9] M. Farber and V. Turaev, *Poincaré-Reidemeister metric, Euler structures, and torsion*, J. Reine Angew. Math. **520** (2000), 195–225.

[10] W. Franz, *Über die torsion einer überdeckung.*, Journal für die reine und angewandte Mathematik **173** (1935), 245–254.

[11] P. B. Gilkey, *The eta invariant and secondary characteristic classes of locally flat bundles*, Algebraic and differential topology—global differential geometry, Teubner-Texte Math., vol. 70, Teubner, Leipzig, 1984, pp. 49–87.

[12] W. Goldman and J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96.

[13] N.-K. Ho, *The real locus of an involution map on the moduli space of flat connections on a Riemann surface*, Int. Math. Res. Not. (2004), no. 61, 3263–3285.

[14] F. Kamber and Ph. Tondeur, *Flat bundles and characteristic classes of group-representations*, Amer. J. Math. **89** (1967), 857–886.

[15] X. Ma and W. Zhang, *η-invariant and flat vector bundles*, Chinese Ann. Math. Ser. B **27** (2006), 67–72.

[16] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), 233–305.

[17] ———, *Analytic torsion and R-torsion for unimodular representation*, Jour. of AMS **6** (1993), 721–753.

[18] R. Ponge, *Spectral asymmetry, zeta functions, and the noncommutative residue*, Internat. J. Math. **17** (2006), 1065–1090.

[19] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Funct. Anal. Appl. **14** (1985), 31–34.

[20] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. **7** (1971), 145–210.

[21] K. Reidemeister, *Homotopieringe und Linsenraume*, Hamburger Abhandl. **11** (1935), 102–109.

[22] R. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure and Appl. Math. AMS **10** (1967), 288–307.

[23] B. V. Shabat, *Introduction to complex analysis. Part II*, Translations of Mathematical Monographs, vol. 110, American Mathematical Society, Providence, RI, 1992, Functions of several variables, Translated from the third (1985) Russian edition by J. S. Joel.

[24] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer Verlag, Berlin, New York, 1987.

[25] V. G. Turaev, *Reidemeister torsion in knot theory*, Russian Math. Survey **41** (1986), 119–182.

[26] ———, *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Math. USSR Izvestia **34** (1990), 627–662.
[27] ———, *Introduction to combinatorial torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001, Notes taken by Felix Schlenk.

Department of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail address: `maximbraverman@neu.edu`
URL: `www.math.neu.edu/~braverman/`

Mathematisches Institut, Universität Bonn, 53115 Bonn, Germany
E-mail address: `vertman@math.uni-bonn.de`
URL: `www.math.uni-bonn.de/people/vertman`